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COMMENT

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Abstract. The structure of finite groups associated with Clifford algebras of signatures $(0, d)$, $d \equiv 3 \pmod 4$, is outlined. A speculation (by another author) concerning the possible exceptional nature of the group associated with $d = 7$ is seen to be false.

In a 1988 paper, Shaw [1] describes a group of order 128 which is associated with $d = 7$ Clifford algebras. In a concluding remark to his paper, he conjectures that the $d = 7$ Clifford algebra is exceptional in that the group of order 128 possesses a maximal Abelian normal subgroup of order 16 consisting entirely of involutions. In this comment, we establish that Shaw's conjecture is false by showing that Clifford algebras for other values of d have associated groups which satisfy this requirement. A detailed description of the groups associated with the Clifford algebras is given by Salingaros [2].

The group of order 2^d has a general presentation which can be written as follows:

$$\langle a, x_1, x_2, \ldots, x_d : a^2 = 1,\ a \text{ is central},\ x_i^2 = a,\ [x_j, x_i] = a,\ x_1 x_2 \ldots x_d = 1\rangle. \qquad (1)$$

Using this general presentation, it is clear that the commutator subgroup, the centre, and the Frattini subgroup of the group are equal. In addition, the centre is cyclic, having order 2. Such a group is called extra-special and has order 2^{2m+1}.

The relevant known results on extra-special 2-groups are summarised below; their proofs may be found in Huppert ([3], pp 349ff). Let G be an extra-special 2-group of order 2^{2m+1}.

Property 1. There are only two isomorphism types of extra-special groups of order 2^{2m+1}. Either

(i) G is a central product of m copies of the dihedral group of order 8, denoted by D_8; or

(ii) G is a central product of $m-1$ copies of D_8 and one copy of the quaternion group of order 8, denoted by Q_8.

Property 2. If G is of type (i), then the maximal Abelian normal subgroups of G have type

$$(\underbrace{1, 1, \ldots, 1}_{m+1}) \quad \text{and} \quad (2, \underbrace{1, \ldots, 1}_{m-1}).$$

Here, an Abelian 2-group of type (a, b) is $C_{2^a} \times C_{2^b}$.

Property 3. If G is of type (ii), then G has maximal Abelian normal subgroups only of type

$$(2, \underbrace{1, \ldots, 1}_{m-1}).$$

In both cases, the subgroups can be described and constructed with relative ease. For example, in a group of type (i) where each dihedral group is generated by a_i and b_i, a maximal Abelian normal subgroup of order 2^{m+1} and consisting entirely of involutions is generated by the centre of the group together with each of the a_i.

Since the groups defined by equation (1) are extra-special, they have isomorphism type (i) or (ii). A non-central element of such a group can be written as $x_{i_1}x_{i_2}\ldots x_{i_l}^{\pm 1}$. The element, w, has length l and its order is either 2 or 4. The number of occurrences of a in $w * w$ is $l(l+1)/2$. If $l(l+1) \bmod 4 = 0$, then w has order 2; otherwise if $l(l+1) \bmod 4 = 2$, w has order 4.

In the groups of interest to Shaw, $d = 4k+3$ where $k \geqslant 0$. The presence of elements of length $(d-1)/2$ and order 4 ensures that there is a single copy of Q_8, thereby ensuring that the central product is of type (ii); if the elements of length $(d-1)/2$ have order 2, then the group is of type (i) and in these cases there are maximal Abelian normal subgroups of order $2^{d+1/2}$ consisting entirely of involutions. The elements of length $(d-1)/2$ have order 4 if $(d-1)/2 \bmod 4 = 1$ while these elements have order 2 if $(d-1)/2 \bmod 4 = 3$.

In summary, for $d = 8k+7$ where $k \geqslant 0$, the group of order 2^d is of type (i) and has maximal Abelian normal subgroups consisting entirely of involutions. For $d = 8k+3$, the group of order 2^d is of type (ii) and has no maximal Abelian normal subgroup consisting entirely of involutions.

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References

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